

Critique of q -entropy for thermal statistics

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During the past 12 years there have been numerous papers on a relation between entropy and probability which is nonadditive and has a parameter q that depends on the nature of the thermodynamic system under consideration. For $q=1$ this relation corresponds to the Boltzmann-Gibbs entropy, but for other values of q it is claimed that it leads to a formalism that is consistent with the laws of thermodynamics. However, it is shown here that the joint entropy for systems having *different* values of q is not defined in this formalism, and consequently fundamental thermodynamic concepts such as temperature and heat exchange cannot be considered for such systems. Moreover, for $q \neq 1$ the probability distribution for weakly interacting systems does not factor into the product of the probability distribution for the separate systems, leading to spurious correlations and other unphysical consequences, e.g., nonextensive energy, that have been ignored in various applications given in the literature.

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I. INTRODUCTION

In 1988 a relation between entropy and probability for thermal statistics was proposed by Tsallis [1], which is non-additive and depends on a parameter q that is presumably determined by the nature of the thermodynamic system under consideration. For the special case $q=1$, this relation reduces to the Boltzmann-Gibbs entropy, but for other values of q it is claimed that this q -entropy leads to an alternative formalism that is “entirely consistent” with the laws of thermodynamics [1–6]. It will be shown, however, that this claim is not valid, because the total entropy of thermodynamic systems that are in thermal contact is not even defined in this formalism for the case where systems have q -entropies with *different* values of q . As a consequence, the fundamental concepts of temperature and heat exchange between such systems cannot be introduced in this formulation of thermodynamics unless q is a universal constant. But even in this case, maximizing the q -entropy of the combined systems leads to unphysical results unless $q=1$. In particular, for $q \neq 1$ the resulting joint probability for states of weakly coupled systems is not the product of the individual probabilities, and as a consequence the total energy of the combined systems is not the sum of the mean energies of each system, although the additivity property is assumed to be satisfied by the individual microstates. This nonadditive property of the mean energy in the q -entropy formalism is manifestly incorrect, and not surprisingly leads to unphysical results, but the consequences have been ignored in various applications of q -entropy in the literature, as will be illustrated in some of the examples discussed here.

The definition of q -entropy for a thermodynamic system with microstates labeled by index i is given by [1–4]

$$S_q = k \frac{\left(1 - \sum_i p_i^q\right)}{(q-1)}, \quad (1)$$

where k is a constant, q is an undetermined parameter, and the quantities p_i are positive numbers that satisfy the condition $\sum_i p_i = 1$. In the limit $q \rightarrow 1$, one recovers the Boltzmann-Gibbs form of the entropy

$$S_1 = -k_B \sum_i p_i \ln(p_i), \quad (2)$$

where $k = k_B$ is the Boltzmann constant. In this special case, p_i is the probability for the occurrence of the i th microstate, and this identification has been extended to the case $q \neq 1$. This extension, however, is not valid as can be seen from the definition of mean values for physical quantities associated with the q -entropy. For example, the internal energy U_q is given by the form

$$U_q = \sum P_i \epsilon_i, \quad (3)$$

where ϵ_i corresponds to the i th energy eigenvalue of the system and $P_i = p_i^q / \sum_j p_j^q$. The quantities P_i are called “escort” probabilities in the q -entropy literature, but according to the conventional definition of mean value in statistics, these P_i ’s are the actual probabilities for the states of the system. Hence for $q \neq 1$ the p_i ’s introduced in the definition of q -entropy, Eq. (1), are devoid of any physical meaning, and are just functions of the probabilities P_j according to the relation

$$p_i = P_i^{1/q} / \left(\sum_j P_j^{1/q} \right). \quad (4)$$

For values of $q \neq 1$, the q -entropy expression introduced in Eq. (1) is shown to be nonadditive by the following arguments [1–4]. Suppose that two thermodynamic systems A and B are weakly coupled or are the subsystems of a larger

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system, and *assume* that the joint probabilities for the states of the combined system are the products of the probabilities for the states of the individual systems. As we shall see later on, this fundamental factorization property is not satisfied by the q -entropy formalism, but surprisingly this fact has been ignored in the literature. According to Eq. (4), we then have

$$p_{i,j}^{AB} = p_i^A p_j^B. \quad (5)$$

Substituting this form into the expression for q -entropy, Eq. (1), one obtains the relation

$$S_q(AB) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)/k. \quad (6)$$

However, this nonadditive relation for the q -entropy leads immediately to a difficulty in the interpretation of S_q as an expression for the thermodynamic entropy [7]. Since weak coupling means that the energy eigenvalues of the combined systems are essentially additive, we have

$$\epsilon_{i,j}^{AB} = \epsilon_i^A + \epsilon_j^B, \quad (7)$$

and, according to Eqs. (3) and (5), the total mean energy $U_q(AB)$ of the combined system is also additive:

$$U_q(AB) = U_q(A) + U_q(B). \quad (8)$$

Assume now that the combined system is isolated, while there is an infinitesimal exchange of energy between systems A and B . Then the variations $\delta S_q(AB) = 0$ and $\delta U_q(AB) = 0$, which implies that

$$\frac{\delta S_q(A)}{1 + (1-q)S_q(A)/k} = - \frac{\delta S_q(B)}{1 + (1-q)S_q(B)/k} \quad (9)$$

and

$$\delta U_q(A) = - \delta U_q(B). \quad (10)$$

Combining these two equations, one finds that

$$[1 + (1-q)S_q(A)/k]T(A) = [1 + (1-q)S_q(B)/k]T(B), \quad (11)$$

where $T(A)$ and $T(B)$ are the absolute temperatures defined by the standard thermodynamic relation

$$\frac{\partial S}{\partial U} = \frac{1}{T}. \quad (12)$$

For two systems in thermal contact, these two temperatures should be equal, but according to Eq. (11) this applies only if $q = 1$. This problem is not unexpected, because temperature should be an *intensive* quantity, but this is not possible in a formalism where the energy is additive, Eq. (8), while the entropy does not satisfy this property. To avoid this problem, it has been proposed [4,8–10] to redefine absolute temperature as the quantity

$$T_q = [1 + (1-q)S_q/k]T, \quad (13)$$

in which case the condition for thermal equilibrium, $T_q(A) = T_q(B)$, is satisfied by Eq. (11). But this definition of temperature, which must be universal, cannot be extended to systems A and B that are described by q -entropies with *different* values q_A and q_B . In this case the q -entropy of the combined system, which is characterized by the quantities $p_{i,j}^{AB}$, Eq. (5), is not defined in terms of these two q parameters. Following the q -entropy formalism, one would have to introduce a new parameter q' for the q -entropy and energy of the combined system, but then these thermodynamic variables cannot be expanded in terms of the corresponding variables for the component systems A and B [11,12]. Instead, according to Eq. (6), the q -entropy of the combined system would be given in terms of *pseudo- q* -entropies for systems A and B with the *same* new parameter q' , and a similar problem would occur with the expansion of the total energy, Eq. (8), i.e.,

$$S_{q'}(AB) = S_{q'}(A) + S_{q'}(B) - (1-q')S_{q'}(A)S_{q'}(B)/k \quad (14)$$

and

$$U_{q'}(AB) = U_{q'}(A) + U_{q'}(B). \quad (15)$$

Consequently, the concepts of thermal equilibrium, temperature, and heat exchange cannot be formulated for such systems. For example, no meaning can be attached to the statement that a system described by the Boltzmann-Gibbs entropy is in thermal equilibrium with a system described by q -entropy with $q \neq 1$. In other words, a Boltzmann-Gibbs thermometer would not be able to measure the temperature of a q -entropic system, and the laws of thermodynamics would therefore fail to have general validity [13].

It follows that the parameter q must be a *universal* constant, just like the Boltzmann constant k , which is applicable to *all* systems in thermodynamic equilibrium. If q is universal, a thermodynamic formulation for an infinitesimal reversible transfer of heat dQ can be given between systems A and B , with

$$dQ = T(A)dS_q(A) = -T(B)dS_q(B), \quad (16)$$

corresponding to an exchange $dU_q(A) = -dU_q(B)$ in the internal energy of these systems. But one is faced with the problem that in this case the temperature $T(A)$ is not equal to $T(B)$, which violates a fundamental principle of thermodynamics for systems in thermal equilibrium. Moreover, the corresponding differentials $T_q(A)dS_q(A)$ and $T_q(B)dS_q(B)$ associated with the proposed redefinition of absolute temperature, Eq. (13), do not have any physical significance. In principle, this problem can be solved [14] by introducing a different form S_q^R for the q -entropy in the thermodynamic relation for the temperature T_q , so that

$$\frac{\partial S_q^R}{\partial U_q} = \frac{1}{T_q}. \quad (17)$$

From the definition of T_q given by Eq. (13) it follows that

$$S_q^R = \frac{k}{(1-q)} \ln[1 + (1-q)S_q]. \quad (18)$$

This alternative expression for the q -entropy was introduced by Rènyi [15] in the form

$$S_q^R = \frac{k}{(1-q)} \ln\left(\sum_i p_i^q\right), \quad (19)$$

which is additive [16], as can be verified by substituting for the p_i 's the form in Eq. (5).

However, there are additional problems with either of these two definitions of q -entropy, even when $q_A = q_B = q$. Maximizing such entropy functions subject to the constraint of constant energy, Eq. (3) yields the q -probability distribution [1–4]

$$p_i \propto [1 - (1-q)\beta\delta\epsilon_i]^{1/(1-q)}, \quad (20)$$

where $\delta\epsilon_i = \epsilon_i - U_q$, U_q is the mean energy, Eq. (3), and β is a parameter related to the inverse temperature. It is clear that the corresponding distribution $p_{i,j}^{AB}$ for the combined system AB does not factor into the product $p_i^A p_j^B$ even when the energies of the microstates are additive, Eq. (7), unless $q = 1$. For example, to first order in $q-1$ [17], one finds that

$$p(\epsilon_i^A + \epsilon_j^B) = p(\epsilon_i^A)p(\epsilon_j^B)[1 + (q-1)\beta^2(\delta\epsilon_i^A \delta\epsilon_j^B)]. \quad (21)$$

In the limit $q = 1$, Eq. (20) reduces to the Boltzmann-Gibbs exponential form for the probability [18],

$$p_i \propto \exp(-\epsilon_i/kT), \quad (22)$$

where $\beta = 1/kT$. As is well known, this canonical distribution follows *uniquely* from the factorization requirement that

$$p(\epsilon_i^A)p(\epsilon_j^B) = p(\epsilon_i^A + \epsilon_j^B). \quad (23)$$

The Boltzmann-Gibbs form for the entropy S , Eq. (2), then follows from the assumption that

$$S = \sum_i f(p_i), \quad (24)$$

where the function $f(p)$ is determined uniquely. Applying the thermodynamic definition of temperature, $\partial S/\partial U = 1/T$, and the Boltzmann-Gibbs definition of the probability distribution, Eq. (22), one obtains the relation

$$\sum_i p_i(\epsilon_i - U) \left[\frac{df(p_i)}{dp_i} + k \ln(p_i) \right] = 0. \quad (25)$$

Hence

$$f(p_i) = -k p_i \ln(p_i) \quad (26)$$

provided $f(1) = f(0) = 0$, which corresponds to the requirement that the entropy vanishes at $T = 0$.

II. UNPHYSICAL PROPERTIES RESULTING FROM APPLICATIONS OF q -ENTROPY TO THERMODYNAMIC SYSTEMS

To illustrate the consequences of disregarding such basic considerations, I would like to call attention to some unphysical results, left unmentioned in the literature, that follow from recent applications of q -entropy to some well-known thermodynamic systems. For example, many papers have been published on the application of q -entropy to blackbody radiation [19–26]. After laborious analysis and numerical computations, the authors in Refs. [23–25] find that for $q \neq 1$ there are deviations from the well-known Stefan-Boltzmann law, which states that the radiation energy depends on the fourth power of the temperature [27]. But since Boltzmann derived this result from purely thermodynamical reasoning, without any statistical assumptions about the form of the entropy, it seems at first sight strange that such a deviation can occur in a formalism that is supposed to satisfy the laws of thermodynamics. The explanation is that as a consequence of the nonextensivity property of the q -entropy formalism, the blackbody energy as well as the entropy do not depend linearly on the volume of the cavity, as was originally assumed by Boltzmann. Hence, the q -energy density and the q -entropy density depend on the volume V of the cavity, although there is no comment concerning this unphysical property in any of the q -entropy calculations in Refs. [19–26]. As follows from simple dimensional arguments, the volume dependence must be given by a dimensionless parameter $V/a(T)^3$, where $a(T)$ is a characteristic length that can depend only on the temperature T . From statistical mechanics we learn that $a(T) = (\hbar c/kT)$ in fact corresponds to the mean thermal wavelength of the blackbody photons. Moreover, expansions in a power series of this parameter which have been applied to fit the cosmic background radiation [19–26] are nonsensical, because in this case $a(T)$ is of order 1/10 cm while the cavity volume V has cosmological dimensions. On purely thermodynamic grounds it can also be shown that if the temperature dependence of the blackbody energy density were to have the form $u \propto T^{4+\delta}$, then the Maxwell-Boltzmann relation $p = (1/3)u$ for the thermal radiation pressure p would lead to a power law volume dependence $u \propto V^{\delta/3}$, and correspondingly $U \propto T(T^3 V)^{1+\delta/3}$, in accordance with our previous dimensional argument. Similarly, one finds that $s \propto V^{\delta/3}$ for the entropy density. But unless $\delta = 0$, such a volume dependence is incompatible with Kirchhoff's law which states that the ratio of emissivity to absorption of radiation in the walls of the cavity must be a universal function of the temperature and the frequency of the radiation. These properties are required in order that such a cavity reach thermal equilibrium. Historically, this law was the original basis for the universality properties of blackbody radiation that culminated in Planck's famous derivation. Finally, the application of q -entropy to blackbody radiation for $q \neq 1$ [19–26] violates also the well-known detailed balance relation for the emission and absorp-

tion of radiation deduced by Einstein in 1917, with conjectured transition probabilities which subsequently were derived from quantum electrodynamics.

Actually, the application of the q -entropy formalism to any system that consists of weakly interacting components leads to unphysical properties. For example, for an ideal gas, the energy is found to depend nonlinearly on the number of particles, the gas pressure p is not equal to $2/3$ of the energy density, and there are correlations between the energies of any pair of particles [28–30], contrary to the very well-known result of kinetic gas theory and statistical mechanics. Similarly, for a system of weakly interacting magnetic moments, anomalous dependences are found for the magnetization and susceptibility on the number of spins [31], although claims have been also made that the magnetization reproduces experimental results in certain manganites [32,33]. The reason for the failure of this formalism to give physically sensible results is not hard to see—it is due to the fact that the probability distribution, Eq. (20), for any two components A_1 and A_2 is not the product of the probabilities for the separate components, as would be expected if these components are weakly interacting. Hence, contrary to basic physical principles, this formalism gives rise to spurious correlations among these components of such systems.

Some proponents of q -entropy have argued that this formalism should be considered only for systems for which the Boltzmann-Gibbs thermodynamic formalism supposedly “fails.” Frequently mentioned as candidates are systems with components that interact primarily through long range forces, such as gravitational forces, for which the total energy and entropy are nonextensive. In Nature, such systems correspond to astrophysical objects such as stars and galaxies, but these objects are generally not found at *maximum entropy* and (correspondingly) *uniform* temperature. For example, stars are either evolving slowly in time, like main sequence stars, which have a large temperature gradient from the interior to the surface, and emit thermal radiation (blackbody energy and entropy), or else have reached a degenerate state such as dwarf stars or neutron stars, provided certain mass limits are satisfied [34]. Otherwise, stars eventually explode into supernovas, sometimes leaving remnants that collapse into such degenerate states, or into a black hole (which are states having maximum entropy). The equilibrium properties of stars are obtained by hydrostatic equations supplemented by *local* thermodynamic equations for matter and radiation based on the Boltzmann-Gibbs entropy [35,36]. Rather than being a failure, the Boltzmann-Gibbs statistics has been applied to stellar structure with enormous success. A q -entropy formalism, however, predicts the existence of finite *isothermal* polytropes as the end products of stellar evolution (states of maximum entropy), which fails completely to account for the observed property of stars in our universe. Such q -polytrope solutions have also been discussed as models for galaxies [38–40].

Another example that has been cited as a so-called failure of Boltzmann-Gibbs entropy, because it involves long-range electromagnetic forces, is the divergence of the partition function Z calculated for the bound states of the hydrogen atom [41]. In this case

$$Z = \sum_n \exp(-\epsilon_n/T), \quad (27)$$

where $\epsilon_n = -R/n^2$ are the bound-state energy levels and R is the Rydberg constant. But the divergence of Z , which occurs in this case because the terms of the series approach unity as n becomes large, is related to the growth of the mean radius of the hydrogen atom, which increases as n^2 . Obviously, in a gas of hydrogen atoms in thermal equilibrium, this radius cannot become larger than the mean distance between atoms. Therefore this distance provides an effective cutoff for the applicability of the hydrogen bound-state energy eigenvalues in the partition sum, because for larger values of n these atoms can no longer be treated even approximately as a gas of noninteracting particles. Instead, for these states, the gas must be viewed as a neutral plasma of electrons and protons interacting via long range electromagnetic forces. Hence the q -entropy formalism, which supposedly gives a finite partition function [41], actually fails to account for the correct physics of this problem. These and other failures, e.g., to understand the solar neutrino flux [42,43], in the application of q -entropy to well-known physical systems mirror the inconsistencies that are inherent in a formulation of thermodynamics based on q -entropy, Eq. (1).

III. CONCLUSION

We have shown that a *prerequisite* to have a q -entropy formalism which is consistent with the laws of thermodynamics is that the parameter q must be a *universal* constant, as is the case also with the constant k that corresponds to Boltzmann’s constant for $q=1$. Moreover, for weakly coupled systems the entropy as well as the energy must be additive, which is a condition explicitly violated by the Tsallis entropy, Eq. (1). Although this condition appears to be satisfied by the Rényi q -entropy, Eq. (19), this is actually not the case, because the probability distribution obtained by maximizing this entropy for $q \neq 1$, Eq. (20), does not satisfy the required factorization condition, Eq. (5), which is required for weakly coupled systems. Thus, we have shown that the only value of q consistent with the laws of thermodynamics is $q=1$, which corresponds to the familiar Boltzmann-Gibbs form for the entropy, Eq. (2). Indeed, it has been explicitly demonstrated here that the application of a q -entropy formalism to blackbody radiation, and to other systems with weakly interacting components, leads to unphysical results when $q \neq 1$.

It has been suggested in the literature that q -entropy calculations are useful because they provide an additional parameter q for comparing theory with observations, but this rationale fails to take into account the fact that such calculations would be inconsistent with the fundamental principles of thermodynamics and statistical mechanics. Hence, if a small departure of q from unity is found in a fit to data, as is claimed, for example, in various analyses of the cosmic blackbody radiation [19–26], such a fit cannot provide any physical insight whatsoever into the source or meaning of the deviations.

Finally, we remark that the q -probability distribution, Eq.

(20), which is obtained by maximizing either of the two q -entropy relations, Eqs. (1) and (19), constrained to a fixed pseudo-energy-function, has also been applied to *nonequilibrium* problems. For example, data on turbulence in a pure-electron plasma column [44] and the velocity distribution in turbulent flow [45,46] have been fitted by such a probability function [47,48]. But absolutely no physical justification has been given for applying to nonequilibrium systems a fundamental condition—maximum entropy—which is associated in statistical mechanics with systems in thermal equilibrium. While such a probability distribution can also be obtained from other *ad hoc* assumptions [47,48], its connection to

q -entropy and nonextensivity is completely unfounded. Although the q -probability distribution appears to be a good phenomenological parametrization for some turbulence data, its deduction from physical principles has not been established [49].

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